

HERMITIAN ANALOGUES OF HILBERT'S 17-TH PROBLEM

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ABSTRACT. We pose and discuss several Hermitian analogues of Hilbert's 17-th problem. We survey what is known, offer many explicit examples and some proofs, and give applications to CR geometry. We prove one new algebraic theorem: a non-negative Hermitian symmetric polynomial divides a nonzero squared norm if and only if it is a quotient of squared norms. We also discuss a new example of Putinar-Scheiderer.

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1. INTRODUCTION

Hilbert's 17-th problem asked whether a non-negative polynomial in several real variables must be a sum of squares of rational functions. E. Artin answered the question in the affirmative in 1927, using the Artin-Schreier theory of real fields. Around 1955 A. Robinson gave another proof using model theory. See [PD] and [S] for much more information about Hilbert's problem. See [R1] for references to recent work and results on concrete aspects of Hilbert's problem. See [He] and [HP] for results and applications in the non-commutative setting.

This present paper aims to survey and organize various results that might be called Hermitian or complex variable analogues of Hilbert's problem. We also obtain a striking new result in Theorem 5.3. The results here and their proofs have a rather different flavor from Hilbert's problem; they are connected for example with ideas such as mapping problems in several complex variables and CR geometry, analytic tools such as compact operators and the Bergman projection, and metrics on holomorphic vector bundles. See [CD1], [CD3], [D2], [D3], [D4], [HP], [Q], [TV], and [V] and their references for additional discussion along the lines of this paper. Both the real and complex cases involve subtle aspects of zero-sets and how they are defined. The author modestly hopes that this paper will encourage people to apply the diverse techniques from the real case to the Hermitian case and that the techniques from the Hermitian case will be useful in the real case as well.

The complex numbers are not an ordered field, and hence to consider non-negativity we must restrict to real-valued polynomials. The natural starting point will be *Hermitian symmetric* polynomials in several complex variables; there is a one-to-one correspondence between real-valued polynomials on \mathbf{R}^{2n} and Hermitian symmetric polynomials on $\mathbf{C}^n \times \mathbf{C}^n$. We begin by clarifying this matter.

Let ρ be a real-valued polynomial on \mathbf{R}^{2n} . Call the real variables (x, y) ; setting $x = \frac{z+\bar{w}}{2}$ and $y = \frac{z-\bar{w}}{2i}$ then determines a polynomial r on $\mathbf{C}^n \times \mathbf{C}^n$ defined by

$$r(z, \bar{w}) = \rho\left(\frac{z+\bar{w}}{2}, \frac{z-\bar{w}}{2i}\right). \quad (1)$$

Polynomials such as r satisfy the Hermitian symmetry condition

$$r(z, \bar{w}) = \overline{r(w, \bar{z})}. \quad (2)$$

We say that r is Hermitian symmetric in n variables.

Proposition 1.1. *Let $r : \mathbf{C}^n \times \mathbf{C}^n \rightarrow \mathbf{C}$ be a polynomial in (z, \bar{w}) . The following statements are equivalent:*

- r is Hermitian symmetric. That is, (2) holds for all z, w .
- The function $z \rightarrow r(z, \bar{z})$ is real-valued.
- We can write $r(z, \bar{w}) = \sum_{\alpha, \beta} c_{\alpha\beta} z^\alpha \bar{w}^\beta$ where the matrix of coefficients is Hermitian symmetric: $c_{\alpha\beta} = \overline{c_{\beta\alpha}}$ for all α, β .

Conversely, given a Hermitian symmetric polynomial r , the function $z \rightarrow r(z, \bar{z})$ can be regarded as a polynomial in the real and imaginary parts of z . We express the ideas via Hermitian symmetric polynomials, for several compelling reasons: the role of complex analysis is evident, we can polarize by treating z and \bar{z} as independent variables, and Hermitian symmetry leads to elegance and simplicity not observed in the real setting.

Proposition 1.1 suggests using Hermitian linear algebra to study real-valued polynomials $z \rightarrow r(z, \bar{z})$. The polynomial is Hermitian symmetric if and only if the matrix $C = (c_{\alpha\beta})$ is Hermitian symmetric. On the other hand, the condition that r be *non-negative* as a function is **not** the same as the non-negativity of the matrix C . Much of our work will be firmly based on clarifying this point.

First we introduce a natural concept. We say that $\mathbf{s}(r) = (A, B)$ if C has A positive and B negative eigenvalues. We call (A, B) the *signature pair* of r and $A + B$ the *rank* of r . See [D1] or [D3] for versions and applications of the following basic statement.

Proposition 1.2. *Let $r : \mathbf{C}^n \times \mathbf{C}^n \rightarrow \mathbf{C}$ be a Hermitian symmetric polynomial. Then $\mathbf{s}(r) = (A, B)$ if and only if there are linearly independent holomorphic polynomials $f_1, \dots, f_A, g_1, \dots, g_B$ such that*

$$r(z, \bar{z}) = \sum_{j=1}^A |f_j(z)|^2 - \sum_{j=1}^B |g_j(z)|^2 = \|f(z)\|^2 - \|g(z)\|^2. \quad (3)$$

If $\mathbf{s}(r) = (A, 0)$ for some A (including 0) then we call r a *squared norm*. Squared norms take only non-negative values, but non-negative Hermitian symmetric functions need not be squared norms. In a moment we will give two simple but instructive examples. In order to clarify these examples and to state several analogues of Hilbert's problem, we introduce some of the positivity conditions we will be using. See Section 2 for a detailed discussion of these and several other conditions.

Definition 1.1. Positivity classes of Hermitian symmetric polynomials.

- $\mathcal{P}_1 = \mathcal{P}_1(n)$ denotes the set of non-negative Hermitian symmetric polynomials in n variables.
- $\mathcal{P}_\infty = \mathcal{P}_\infty(n)$ denotes the set of Hermitian symmetric polynomials in n variables that are squared norms of holomorphic polynomial mappings. Thus $r \in \mathcal{P}_\infty$ if and only if $r = \|h\|^2$ for a holomorphic polynomial mapping h .
- $\mathcal{Q} = \mathcal{Q}(n)$ denotes the set of polynomials that are quotients of elements of \mathcal{P}_∞ . Thus $r = \frac{\|F\|^2}{\|G\|^2}$ for holomorphic polynomial mappings F, G .

- $\mathcal{Q}' = \mathcal{Q}'(n)$ denotes the set of $r \in \mathcal{P}_1$ for which there is an $s \in \mathcal{P}_1$ (not identically 0) and a holomorphic polynomial mapping F with $rs = \|F\|^2$.
- $\text{rad}(\mathcal{P}_\infty)$ denotes the set of $r \in \mathcal{P}_1$ for which there is an integer N such that $r^N \in \mathcal{P}_\infty$. Thus $r^N = \|h\|^2$.

The following inclusions are easy to verify:

$$\mathcal{P}_\infty \subset \mathcal{Q} \subset \mathcal{Q}' \subset \mathcal{P}_1. \quad (4)$$

$$\mathcal{P}_\infty \subset \text{rad}(\mathcal{P}_\infty) \subset \mathcal{Q}' \subset \mathcal{P}_1. \quad (5)$$

Most of these inclusions are strict. Here are simple but instructive examples. See also Example 2.1.

Example 1.1. For $\alpha \in \mathbf{R}$, for $n = 1$ and $z = x + iy$, put

$$r(z, \bar{z}) = \alpha(z + \bar{z})^2 + |z|^2 = (1 + 4\alpha)x^2 + y^2. \quad (6)$$

The following statements hold:

- $r \in \mathcal{P}_1$ if and only if $\alpha \geq \frac{-1}{4}$.
- $r \in \mathcal{P}_\infty$ if and only if $\alpha = 0$.
- $r \in \mathcal{Q}$ if and only if $\alpha = 0$.
- $r \in \mathcal{Q}'$ if and only if $\alpha = 0$.

Next, for $\lambda \in \mathbf{R}$, and for $n = 2$, put

$$r(z, \bar{z}) = |z_1|^4 + \lambda|z_1 z_2|^2 + |z_2|^4. \quad (7)$$

The following statements hold:

- $r \in \mathcal{P}_1$ if and only if $\lambda \geq -2$.
- $r \in \mathcal{P}_\infty$ if and only if $\lambda \geq 0$.
- $r \in \mathcal{Q}$ if and only if $\lambda > -2$.
- $r \in \mathcal{Q}'$ if and only if $\lambda > -2$.

Example 1.1 shows that two of the containments in (4) are strict. In Example 2.1 we will see that both containments in (5) are strict. In Section 5 we prove a surprising result:

$$\mathcal{Q} = \mathcal{Q}'. \quad (8)$$

In many instructive examples the coefficients depend on parameters. Let K be a closed subset of \mathbf{R}^k . Suppose for each $\lambda \in K$ that $c_{\alpha\beta}(\lambda)$ is a Hermitian symmetric matrix and that the map $\lambda \rightarrow c_{\alpha\beta}(\lambda)$ is continuous. We consider the family of Hermitian symmetric polynomials r_λ defined for $\lambda \in K$ by

$$r_\lambda(z, \bar{w}) = \sum c_{\alpha\beta}(\lambda) z^\alpha \bar{w}^\beta.$$

Let \mathcal{S} be a set of Hermitian symmetric polynomials. We say that \mathcal{S} is *closed under limits* if, whenever $r_\lambda \in \mathcal{S}$ and $\lim(\lambda) = L$, then $r_L \in \mathcal{S}$. By Example 1.1, \mathcal{Q} is not closed under limits. It is however closed under sum and product.

In addition to determining which of the containments are strict, we would like to provide alternative characterizations of the various sets. For example, two separate results mentioned in Remark 2.1 each characterize \mathcal{P}_∞ . These remarks therefore suggest the following analogues of Hilbert's problem. We discuss answers to Analogue 1 from [V], [D4], [D5]. One new aspect of this paper is the introduction and analysis of \mathcal{Q}' and Analogue 2. Theorem 5.3 states that $\mathcal{Q}(n) = \mathcal{Q}'(n)$ for all n and hence answers Analogue 2. So far Analogue 3 has no nice answer. See Section 7 for

some results. See [DP] and the discussion near Example 3.1 for more on Analogue 4. Below we pose additional questions related to all these analogues.

Analogue 1. Give tractable necessary and sufficient conditions for a polynomial to lie in \mathcal{Q} .

Analogue 2. Give tractable necessary and sufficient conditions for a polynomial to lie in \mathcal{Q}' .

Analogue 3. Give tractable necessary and sufficient conditions for a polynomial to lie in $\text{rad}(\mathcal{P}_\infty)$.

Analogue 4. Generalize the discussion to algebraic sets and ideals. For example, if a polynomial is positive on an algebraic set, must it agree with a squared norm there?

Similarities and differences between the real and complex cases. We pursue the analogy with Hilbert's problem and discover some significant differences.

In the real case, after putting everything over a common denominator, we can state Artin's theorem as follows. A real polynomial r is non-negative if and only if there is a polynomial q such that $q^2 r$ is a sum of squares of polynomials. Thus

$$q^2 r = \sum p_j^2 = \|p\|^2. \quad (9.1)$$

In the complex case, let $r = \frac{\|f\|^2}{\|g\|^2}$ be a quotient of squared norms. Then we have

$$\|g\|^2 r = \|f\|^2. \quad (9.2)$$

In both cases we can regard the denominator as a multiplier to bring us into the good situation of squared norms. Notice however a difference between (9.1) and (9.2). In (9.1) it suffices for the multiplier to be the square of a single polynomial. In (9.2), even by allowing the rank of $\|g\|^2$ to be arbitrarily large, we still do not get all non-negative Hermitian symmetric r . Hence we naturally allow the possibility

$$sr = \|f\|^2, \quad (9.3)$$

where s is an arbitrary non-negative Hermitian symmetric polynomial. We still do not get all non-negative polynomials r in this way.

In both the real and complex cases we naturally seek the minimum number of terms required in the sums on the right-hand sides. A famous result of Pfister [Pf] says in n real dimensions that 2^n terms suffice; this result is remarkable for two reasons. First, it is independent of the degree. Second, despite considerable work, it is unknown what the sharp bound is. We sound one warning. For $n \geq 2$ there exist non-negative polynomials in n variables that cannot be written as sums of squares with 2^n terms. This statement does not contradict Pfister's result, which says after multiplication by some q^2 that the product can be written as a sum of squares with at most 2^n terms.

In the complex case, when r satisfies (9.3) we seek the minimum possible rank of $\|f\|^2$ and when r satisfies (9.2) we also seek the minimum possible rank for $\|g\|^2$. No bounds exist depending on only the dimension and the degree of r . If $n = 1$ and $r(z, \bar{z}) = (1 + |z|^2)^d$, then the rank of r is $d + 1$, which obviously depends on the degree. One cannot write r (or any non-zero multiple of r) as a squared norm with fewer than $d + 1$ terms. Hence the analogue of Pfister's result fails. The warning above suggests that we must consider the possibility of rank dropping under

multiplication, and thus motivates Section 9. Proposition 9.1 gives an example of maximal collapse in rank.

Question 1. Assume $r \in \mathcal{Q}'$. What is the minimum rank of any non-zero squared norm $\|f\|^2$ divisible by r ?

Consider the polynomial r_λ defined in (7). When $\lambda > -2$, r_λ is a quotient $\frac{\|f_\lambda\|^2}{\|g_\lambda\|^2}$ of squared norms and specific maps f_λ and g_λ are known. The ranks of these squared norms both tend to infinity as λ tends to -2 . When $\lambda = -2$, r is not even in \mathcal{Q}' . The reason is that its zero-set is not contained in any complex algebraic variety of positive codimension. See the discussion following Definition 1.2.

See Definition 2.3 for the meaning of bihomogeneous. We will pass back and forth between arbitrary Hermitian polynomials and bihomogeneous ones.

The paper [S] discusses situations in the real setting regarding sums of squares where one must carefully distinguish between positivity and non-negativity. Zero-sets matter. Our analogues of Hilbert's problem also involve subtle issues about zero-sets. For example, the main result in [Q] or [CD1] (see Theorem 3.1) implies the following. If r is bihomogeneous and strictly positive away from the origin, then $r \in \mathcal{Q}$. On the other hand, put $r(z, \bar{z}) = (|z_1|^2 - |z_2|^2)^2$; thus $\lambda = -2$ in (7). Then r is bihomogeneous but it vanishes along the wrong kind of set for it to divide a squared norm (except 0). Let $\mathbf{V}(r)$ denote the zero-set of r . For r to be in \mathcal{Q} or \mathcal{Q}' , not only must $\mathbf{V}(r)$ be a complex variety, but r must define it correctly.

Definition 1.2. A non-negative Hermitian polynomial r has a *properly defined* zero-set if there is a holomorphic polynomial mapping h , $\epsilon > 0$, and a Hermitian polynomial s such that $s \geq \epsilon > 0$ for which $r = \|h\|^2 s$.

Example 1.2. For $n = 1$ and $\alpha \geq 0$ put $r(z, \bar{z}) = \alpha|z|^2 + (z + \bar{z})^2$. If $\alpha > 0$, then $\mathbf{V}(r) = \{0\}$, but r defines 0 in the wrong way. When $\alpha = 0$, $\mathbf{V}(r)$ is even worse; it is the line given by $x = 0$. In either case, r is not in \mathcal{Q}' . The polynomial $1 + r$ is also not in \mathcal{Q}' . See Theorem 4.1.

Thus, if $r \in \mathcal{Q}'$, then r has a properly defined zero-set. By Example 1.2, the converse fails, even when r is strictly positive. We also must be careful because there exist positive polynomials whose infima are zero. See Example 4.1.

Suppose $r(z, \bar{z})$ is divisible (as a polynomial) by $\|h(z)\|^2$ for a non-constant holomorphic polynomial mapping h . Then $\mathbf{V}(r)$ contains the complex variety defined by h . By Lemma 2.4, $r \in \mathcal{Q}$ if and only if $\frac{r}{\|h\|^2} \in \mathcal{Q}$. The same statement holds with \mathcal{Q} replaced by \mathcal{Q}' . There is no loss in generality if we therefore assume that all such factors have been canceled. The result might still have zeroes and hence cause trouble.

We briefly return to the holomorphic decomposition (3) of a Hermitian symmetric polynomial. Let r be a bihomogeneous Hermitian symmetric polynomial, and assume r is not identically 0. If $r \in \mathcal{P}_\infty$, then we may write $r = \|f\|^2$, where the components of f are linearly independent. By [D1], f is determined up to a unitary transformation. We regard this situation as understood.

Suppose next that $r \in \mathcal{P}_1$ but r is not in \mathcal{P}_∞ . We write $r = \|f\|^2 - \|g\|^2$ where $g \neq 0$ and the components of f and g form a linearly independent set. Consider, for each $\lambda \in \mathbf{R}$, the family r_λ defined by

$$r_\lambda(z, \bar{z}) = \|f(z)\|^2 - \lambda \|g(z)\|^2. \quad (10)$$

For $\lambda \leq 0$ it is obvious that $r_\lambda \in \mathcal{P}_\infty$, and this case is understood. For $0 \leq \lambda \leq 1$, r_λ defines a homotopy between $\|f\|^2$ and r .

Varolin's solution to Analogue 1, although expressed in different language in [V], amounts to saying (after dividing out factors of the form $|h|^2$) that $r \in \mathcal{Q}$ if and only if there is a $\lambda > 1$ such that $r_\lambda \in \mathcal{P}_1$. (Equivalently, if there is a constant $c < 1$ such that $\|g\|^2 \leq c\|f\|^2$.) Varolin works in the bihomogeneous setting and even more generally with Hermitian combinations of sections of certain line bundles over compact complex manifolds. Note that a homogeneous polynomial may be regarded as a section of a power of the hyperplane bundle over projective space. We briefly discuss such considerations in Section 8.

Varolin's proof uses the resolution of singularities to reduce to bihomogeneous polynomials in two complex variables. Dehomogenizing then reduces to the case of one complex dimension, where the problem was solved in [D4]. We give an improved treatment of that work in Section 4. The proof in one dimension relies on the result in a nondegenerate situation in two dimensions, and we present that information in Section 3. Zero-sets also play a crucial part in Varolin's approach. Our definition of properly defined zero set differs slightly from his concept of *basic zeroes*.

Our result that $\mathcal{Q} = \mathcal{Q}'$ solves Analogue 2. By contrast, the author knows of no satisfactory answer to Analogue 3. See Section 7 for some information.

Analogue 4 has not yet been fully studied. Theorem 3.3 shows that a Hermitian polynomial positive on the unit sphere agrees with a squared norm there. Example 3.1 shows that there exist algebraic strongly pseudoconvex hypersurfaces X and Hermitian polynomials f such that $f > 0$ on X but f agrees with no squared norm there. One needs analogues of the \mathcal{P}_k spaces for real polynomial ideals; see [DP] for recent work in this direction.

Signature pairs. We close the introduction by considering signature pairs. By (9.2) we see that $r \in \mathcal{Q}$ if and only if there is a multiplier $\|g\|^2 = q \in \mathcal{P}_\infty$ such that $qr \in \mathcal{P}_\infty$. In particular, suppose $\mathbf{s}(r) = (A, B)$. Then there is a q with $\mathbf{s}(q) = (N_1, 0)$ such that $\mathbf{s}(qr) = (N_2, 0)$. The integers N_1 and N_2 are complex variable analogues of the number of terms required in the sums of squares of polynomials from the real variable setting.

We thus study the behavior of the signature pair under multiplication in analyzing complex variable analogues of Hilbert's problem. The papers [DL] and [G] apply results about signature pairs to CR Geometry. In particular Grundmeier [G] computes the signature pair for various group-invariant Hermitian symmetric polynomials that determine invariant holomorphic polynomial mappings from spheres to hyperquadrics.

By definition $R \in \mathcal{P}_\infty$ if and only if its signature pair is $(N, 0)$ for some N . The following question generalizes both Analogue 2 and Question 1.

Question 2. Suppose R is Hermitian symmetric and that it factors: $R = qr$. What can we say about the relationships among the signature pairs for q, r, R ?

Some partial answers to this question appear in [DL]. There exist pairs of (quite special) Hermitian symmetric polynomials with arbitrarily large ranks whose product has signature pair $(2, 0)$ and hence rank 2. We call this phenomenon *collapsing of rank*. This collapse is sharp, in the sense that we cannot obtain rank 1. A similar result fails for real-analytic Hermitian symmetric functions.

We also note the following fact, which is applied in [DL]. Given a pair (A, B) with $A + B \geq 2$, there exist Hermitian polynomials r_1 and r_2 such that all the entries in the signature pairs (A_j, B_j) are positive, but yet the signature pair of the product is (A, B) .

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2. POSITIVITY CONDITIONS

In this section we define various positivity conditions and introduce notation for them. The definitions include several notions not discussed in the introduction. We assume that the dimension n is fixed and do not include it in the notation.

Definition 2.1. Positivity conditions for Hermitian symmetric polynomials.

- 1) $r \in \mathcal{P}_1$ if $r(z, \bar{z}) \geq 0$ for all z .
- 2) For $k \in \mathbf{N}$, we say that $r \in \mathcal{P}_k$ if, for every choice of k points z_1, \dots, z_k in \mathbf{C}^n , the matrix $r(z_i, \bar{z}_j)$ is non-negative definite.
- 3) $r \in \mathcal{P}_\infty$ if there is a holomorphic polynomial mapping f such that

$$r(z, \bar{z}) = \|f(z)\|^2. \quad (11)$$

By Proposition 1.1 and linear algebra, (11) holds if and only if the underlying matrix C of Taylor coefficients of r is of the form A^*A .

- 4) $r \in \mathcal{Q}$ if r is the quotient of elements of \mathcal{P}_∞ . In other words, there are holomorphic polynomial mappings F and G such that

$$r(z, \bar{z}) = \frac{\|F(z)\|^2}{\|G(z)\|^2}. \quad (12)$$

- 5) $r \in \mathcal{Q}'$ if r is in \mathcal{P}_1 and there is $s \in \mathcal{P}_1$ (not identically 0) and $\|F\|^2 \in \mathcal{P}_\infty$ such that $rs = \|F\|^2$.
- 6) $r \in \text{rad}(\mathcal{P}_\infty)$ if $r \geq 0$ and there is a positive integer N such that $r^N \in \mathcal{P}_\infty$.
- 7) r satisfies the global Cauchy-Schwarz inequality if, for all z and w ,

$$r(z, \bar{z})r(w, \bar{w}) \geq |r(z, \bar{w})|^2. \quad (13)$$

If (13) holds, then r achieves only one sign, and $|r| \in \mathcal{P}_1$.

- 8) $r \in \mathcal{L}$ if $r \geq 0$ and $\log(r)$ is plurisubharmonic.

Remark 2.1. It is well-known, and proved for example in [AM] and [DV], that $r \in \mathcal{P}_k$ for all k if and only if $r \in \mathcal{P}_\infty$. Thus

$$\mathcal{P}_\infty = \bigcap_{j=1}^\infty \mathcal{P}_j, \quad (14)$$

and (14) gives an alternative definition of \mathcal{P}_∞ .

We mention a second characterization of squared norms from [HP]. Given $r(z, \bar{z})$, replace each variable z_j by a matrix Z_j (of arbitrary size) and replace \bar{z}_j by the adjoint Z_j^* to obtain $r(Z, Z^*)$. Then $r \in \mathcal{P}_\infty$ if and only if, for all *commuting* n -tuples $Z = (Z_1, \dots, Z_n)$, we have $r(Z, Z^*) \geq 0$. (All such matrices are non-negative definite.)

Remark 2.2. Inequality (13) is a curvature condition; it arises when r is regarded as a (possibly degenerate) metric on a holomorphic line bundle. See [Cal], [CD3], [D2], and especially [V].

From the definitions we immediately see that $\mathcal{P}_{k+1} \subset \mathcal{P}_k$ for all k . Examples from [DV] show that the classes \mathcal{P}_k are distinct. On the other hand, if the degrees of polynomials under consideration are bounded, then there is a k_0 such that the classes \mathcal{P}_k are the same for $k \geq k_0$. We recall a concept from [DV].

Definition 2.2. (Stability Index) Let S be a subset of \mathcal{P}_1 . We define $I(S)$ to be the smallest k for which

$$S \cap \mathcal{P}_\infty = S \cap \mathcal{P}_k.$$

If no such k exists we write $I(S) = \infty$. When $I(S)$ is finite we say that S is *stable*.

The stability index is computed in [DV] in several interesting situations. From that work it follows that sets of Hermitian symmetric polynomials of bounded degree are stable. In other words, only finitely many of the sets \mathcal{P}_k are distinct if we fix the dimension and bound the degree.

The author does not know if there are stability criteria for the result of [HP] mentioned in Remark 2.1 or the main result in [He]. It seems however that results in this direction could be quite useful.

Remark 2.3. For each subset \mathcal{P}_k there is a corresponding sharp version; we demand that the matrix $R(z_i, \bar{z}_j)$ be positive definite whenever the points are distinct.

Recall that \mathcal{S} is *closed under limits* if, whenever $r_\lambda \in \mathcal{S}$ and $\lim(\lambda) = L$, then $r_L \in \mathcal{S}$. By Example 1.1, \mathcal{Q} is not closed under limits. It is however closed under sum and product. For each k it is evident that \mathcal{P}_k is closed under limits. These sets are also closed under sum and product. See Lemma 2.4.

The following example offers some insight into the relationships among the various conditions.

Example 2.1 (DV). Consider the family of polynomials r_λ given by

$$r_\lambda(z, \bar{z}) = (|z_1|^2 + |z_2|^2)^4 - \lambda |z_1 z_2|^4. \quad (15)$$

The following hold:

- $r_\lambda \in \mathcal{P}_1$ if and only if $\lambda \leq 16$.
- $r_\lambda \in \mathcal{Q}$ if and only if $r_\lambda \in \mathcal{Q}'$ if and only if $\lambda < 16$.
- $r_\lambda \in \mathcal{L}$ if and only if $\lambda \leq 12$.
- $r_\lambda \in \mathcal{P}_2$ if and only if $\lambda \leq 8$.
- $r_\lambda \in \text{rad}(\mathcal{P}_\infty)$ if and only if $\lambda < 8$.
- For $k > 2$, $r_\lambda \in \mathcal{P}_k$ if and only if $r_\lambda \in \mathcal{P}_\infty$.
- $r_\lambda \in \mathcal{P}_\infty$ if and only if $\lambda \leq 6$.

Remark 2.4. The corresponding function in n dimensions is

$$r_a(z, \bar{z}) = \|z\|^{4n} - a \left| \prod z_j \right|^4.$$

Then $r \in \mathcal{P}_1$ for $a \leq n^{2n}$, the Cauchy-Schwarz inequality (13) fails for $a > \frac{n^{2n}}{2}$, and $r \in \mathcal{P}_\infty$ for $a \leq \frac{(2n)!}{2^n}$.

A sharp form of inequality (13) arises in the isometric embedding theorem from [CD3]. If $r \geq 0$, then (13) is equivalent to $r \in \mathcal{P}_2$. If $r \in \text{rad}(\mathcal{P}_\infty)$, then r must satisfy (13). Example 1.2 shows that the converse fails. On the other hand, if r satisfies an appropriate sharp form of (13), then $r \in \text{rad}(\mathcal{P}_\infty)$. See Theorem

7.1. For a fixed bound on the degree, the set of polynomials satisfying (13) can be identified with a closed cone in some Euclidean space. Every point in the interior of this cone corresponds to an element of $\text{rad}(\mathcal{P}_\infty)$, but only a proper subset of the boundary points do. Analogue 3) is thus closely related to but distinct from studying \mathcal{P}_2 .

Next we note some obvious properties of the sets \mathcal{P}_k and their analogues for \mathcal{Q} . We continue by discussing bihomogenization and the surprisingly useful special case when the underlying matrix of a Hermitian symmetric polynomial is diagonal.

Lemma 2.1. *Let $t \rightarrow z(t)$ be a holomorphic polynomial \mathbf{C}^n -valued mapping. Let z^*r denote the pullback mapping $t \rightarrow r(z(t), \overline{z(t)})$. The following hold:*

- $r \in \mathcal{Q}(n)$ implies $z^*r \in \mathcal{Q}(1)$.
- $r \in \mathcal{Q}'(n)$ implies $z^*r \in \mathcal{Q}'(1)$.
- For $k \geq 1$ or $k = \infty$, $r \in \mathcal{P}_k(n)$ implies $z^*r \in \mathcal{P}_k(1)$.

Proof. We omit the proof, as these statements are all easy to check. \square

Let r be a Hermitian symmetric polynomial of degree m in z . Even when $r \geq 0$, its total degree $2d$ can be any even value in the range $m \leq 2d \leq 2m$. For squared norms, however, there is an obvious restriction on $2d$.

Lemma 2.2. *If $r = \|f\|^2 \in \mathcal{P}_\infty$, then the total degree of r is twice the degree of r in z .*

Proof. Write $f = f_0 + \dots + f_d$ as its expansion into homogeneous parts. Regard z and \bar{z} as independent variables. Then r is of degree d in z . Its terms of highest total degree equal $\|f_d\|^2$ and hence the total degree of r is $2d$. \square

Lemma 2.3. *Each \mathcal{P}_k is closed under sum and under product. For each k we have $\mathcal{P}_{k+1} \subset \mathcal{P}_k$. Each \mathcal{P}_k is closed under limits.*

Proof. These facts follow easily from the part of Definition 2.1 giving \mathcal{P}_k . The proof of closure under product uses a well-known lemma of Schur: if (a_{ij}) and (b_{ij}) are non-negative definite matrices of the same size, then their Schur product $(a_{ij}b_{ij})$ is also non-negative definite. See [AM] or [D3]. \square

Lemma 2.4. *\mathcal{Q} is closed under sums and products but not under limits. \mathcal{Q}' is closed under products but not under limits.*

Proof. Suppose $r = \frac{\|f\|^2}{\|g\|^2}$ and $R = \frac{\|F\|^2}{\|G\|^2}$. Then

$$r + R = \frac{\|(f \otimes G) \oplus (g \otimes F)\|^2}{\|g \otimes G\|^2} \quad (16)$$

$$rR = \frac{\|f \otimes F\|^2}{\|g \otimes G\|^2}. \quad (17)$$

For the case of \mathcal{Q}' we assume that $r_j s_j = \|f_j\|^2$ for $j = 1, 2$. Then we have

$$(r_1 r_2)(s_1 s_2) = \|f_1 \otimes f_2\|^2. \quad (18)$$

Formula (7) from Example 1.1 shows that \mathcal{Q} and \mathcal{Q}' are not closed under limits. \square

Definition 2.3. A Hermitian symmetric polynomial r is called *bihomogeneous* of total degree $2m$ if, for all $\lambda \in \mathbf{C}$,

$$r(\lambda w, \overline{\lambda w}) = |\lambda|^{2m} r(w, \overline{w}). \quad (19)$$

For example, $|z|^{2m}$ is bihomogeneous, but $z^k + \bar{z}^k$ is not. Let r be a Hermitian symmetric polynomial on \mathbf{C}^n , and assume r is of degree m in z . (Its total degree lies in the interval $[m, 2m]$.) We can bihomogenize r by adding the variable $t = z_{n+1}$ and its conjugate. Its bihomogenization Hr is defined for $t \neq 0$ by

$$(Hr)(z, t, \bar{z}, \bar{t}) = |t|^{2m} r\left(\frac{z}{t}, \frac{\bar{z}}{\bar{t}}\right) \quad (20)$$

and by continuity at $t = 0$. It is evident that if $w = (z, t)$ and $\lambda \in \mathbf{C}$, then (19) holds for Hr . We say that Hr is bihomogeneous of total degree $2m$.

For any k , $r \in \mathcal{P}_k(n)$ if and only if $Hr \in \mathcal{P}_k(n+1)$. Furthermore $r \in \mathcal{Q}(n)$ if and only if $Hr \in \mathcal{Q}(n+1)$. Thus we will often work in the bihomogeneous setting.

3. STABILIZATION IN THE NONDEGENERATE CASE

Let r be a bihomogeneous polynomial that is positive away from zero. In this section we develop the machinery to prove that $r \in \mathcal{Q}$. We give many applications in the rest of the paper.

Let B_n denote the unit ball in \mathbf{C}^n . We denote by $A^2(B_n)$ the Hilbert space of square-integrable holomorphic functions on the ball; it is a closed subspace of $L^2(B_n)$. We write V_m for the complex vector space of homogeneous holomorphic polynomials of degree m . The monomials form a complete orthogonal system for $A^2(B_n)$ and hence V_m is orthogonal to V_d for $m \neq d$.

The *Bergman projection* is the self-adjoint projection $P : L^2(B_n) \rightarrow A^2(B_n)$. The Bergman kernel function for B_n is the Hermitian symmetric real-analytic function $B(z, \bar{w})$ defined for $f \in L^2(B_n)$ by the formula

$$Pf(z) = \int_{B_n} B(z, \bar{w}) f(w) dV(w).$$

It is well-known that

$$B(z, \bar{w}) = \frac{n!}{\pi^n} \frac{1}{(1 - \langle z, w \rangle)^{n+1}}. \quad (21)$$

We will use several facts about P and B . In particular we note that

$$B(z, \bar{w}) = \sum_{j=0}^{\infty} c_j \langle z, w \rangle^j, \quad (22)$$

where each c_j is a positive number.

Lemma 3.1. *Let M be multiplication by a bounded function on $L^2(B_n)$. Then the commutator $[P, M]$ is compact on $L^2(B_n)$.*

Proof. This fact can be directly checked for the ball. See [CD2] for a general result to the effect that compactness estimates for the $\bar{\partial}$ -Neumann problem (well-known for the ball) imply that such a commutator is also compact. See [Str] for a simpler proof and considerable additional information about compactness for the $\bar{\partial}$ -Neumann problem. \square

Note that a power of the squared Euclidean norm is itself a squared norm; $\|z\|^{2d} = \|H_d\|^2$, where H_d is the d -fold symmetric tensor product of the identity map with itself. Observe also that the components of H_d form a basis for V_d .

Let us order in some fashion the multi-indices of degree at most m . A Hermitian symmetric polynomial r then can be considered as the restriction of the Hermitian form in N variables

$$\sum_{\alpha, \beta=1}^N c_{\alpha\beta} \zeta_\alpha \bar{\zeta}_\beta \quad (23)$$

to the Veronese variety given by parametric equations $\zeta_\alpha(z) = z^\alpha$. If r is bihomogeneous of total degree $2m$, then r determines a Hermitian form on V_m via its underlying matrix of coefficients. We will use Hermitian symmetric polynomials as integral kernels of operators on $A^2(B_n)$. Given such an r , we define T_r as follows:

$$(T_r f)(z) = \int_{B_n} r(z, \bar{w}) f(w) dV(w). \quad (24)$$

When r is bihomogeneous of total degree $2m$, T_r annihilates every V_j except V_m . Furthermore we have the following simple lemma.

Lemma 3.2. *Let r be a bihomogeneous polynomial of total degree $2m$. Then $r \in \mathcal{P}_\infty$ if and only if T_r is non-negative definite on V_m . That is $\langle T_r f, f \rangle \geq 0$ for all $f \in V_m$. Here*

$$\langle T_r f, f \rangle = \int_{B_n} \int_{B_n} r(z, \bar{w}) f(w) \overline{f(z)} dV(w) dV(z). \quad (25)$$

Theorem 3.1. *([Q], [CD1]) Let $r(z, \bar{z}) = \sum c_{\alpha\beta} z^\alpha \bar{z}^\beta$ be a bihomogeneous Hermitian symmetric polynomial of total degree $2m$. The following are equivalent:*

- 1) r achieves a positive minimum value on the sphere.
- 2) There is an integer d such that the underlying Hermitian matrix for $\|z\|^{2d} r(z, \bar{z})$ is positive definite. Thus

$$\|z\|^{2d} r(z, \bar{z}) = \sum E_{\mu\nu} z^\mu \bar{z}^\nu \quad (26)$$

where $(E_{\mu\nu})$ is positive definite.

3) Let R_{m+d} be the operator defined by the kernel $k_d(z, \zeta) = \langle z, \zeta \rangle^d r(z, \bar{\zeta})$. There is an integer d such that $R_{m+d} : V_{m+d} \rightarrow V_{m+d}$ is a positive operator.

4) There is an integer d and a holomorphic homogeneous vector-valued polynomial g of degree $m+d$ such that $\mathbf{V}(g) = \{0\}$ and such that $\|z\|^{2d} r(z, \bar{z}) = \|g(z)\|^2$.

5) Write $r(z, \bar{z}) = \|P(z)\|^2 - \|N(z)\|^2$ for holomorphic homogeneous vector-valued polynomials P and N of degree m . Then there is an integer d and a linear transformation L such that the following are true:

- 5.1) $I - L^* L$ is positive semi-definite.
- 5.2) $H_d \otimes N = L(H_d \otimes P)$
- 5.3) $\sqrt{I - L^* L}(H_d \otimes P)$ vanishes only at 0.

Corollary 3.1. *If r is bihomogeneous and positive on the unit sphere, then $r \in \mathcal{Q}$.*

The main assertion that items 1) and 2) are equivalent was proved in 1967 by Quillen. Unaware of that result, Catlin and the author, motivated by trying to prove Theorem 3.3 below, found a different proof. Both proofs use analysis; Quillen uses Gaussian integrals and *a priori* inequalities on all of \mathbf{C}^n , whereas Catlin-D'Angelo use compact operators and the Bergman kernel function on the unit ball B_n . In both approaches it is crucial that distinct monomials are orthogonal. Theorem 3.1

can be reinterpreted and generalized by expressing it as a statement about metrics on holomorphic line bundles. See [CD3], [V], and Section 8.

The minimum integer d is the same in items 2) and 3). On the other hand, the integer d in item 4) could be smaller. For example, if $r(z, \bar{z}) = |z_1|^8 + |z_2|^8$, then item 4) holds for $d = 0$, but we require $d \geq 3$ for $(|z_1|^2 + |z_2|^2)^d r(z, \bar{z})$ to satisfy (26) with $(E_{\mu\nu})$ positive definite.

We include item 5) because its generalization leads to a (somewhat unsatisfying) solution to Analogue 2. Consider replacing H_d by a general holomorphic mapping B . Suppose $r \in \mathcal{Q}$ and put $r = \frac{\|A\|^2}{\|B\|^2}$. Then there is an L such $B \otimes N = L(B \otimes P)$, 5.1) holds, and $A = \sqrt{I - L^*L}(B \otimes P)$. The analogues of conditions 5.1) and 5.2) give in Proposition 3.1 a necessary and sufficient condition for r to be in \mathcal{Q} . In Theorem 3.1 we know what to use for B , namely $z^{\otimes d}$ for sufficiently large d , whereas Proposition 3.1 provides little concrete information on B .

Proposition 3.1. *(An answer to Analogue 1) Suppose $r = \|P\|^2 - \|N\|^2$. Then $r \in \mathcal{Q}$ if and only if there is a holomorphic polynomial mapping B and a linear mapping L such that*

- $I - L^*L$ is non-negative definite.
- $B \otimes N = L(B \otimes P)$.

Next we mention a special case of Theorem 3.1 which goes back to Polya in 1928 and which has many proofs. See for example [D3], [HLP], [R2], and [S]. Let R be a homogeneous polynomial on \mathbf{R}^N . Let $s(x) = \sum_{j=1}^N x_j$, and let H denote the part of the hyperplane defined by $s(x) = 1$ and lying in the first orthant. Reznick [R2] obtains bounds on the integer d in Theorem 3.2 in terms of the dimension N , the degree m of r , and the ratio of the maximum and minimum of R on H . To and Yeung [TY] combine the ideas from [R2] and [CD1] to give estimates on d from Theorem 3.1 in terms of similar information. We emphasize that no bounds involving only the dimension and the degree are possible. The following result is the special case of Theorem 3.1 when $r(z, \bar{z})$ depends on only the variables $|z_1|^2, \dots, |z_N|^2$.

Theorem 3.2. *(Polya) Let $R(x)$ be a real homogeneous polynomial on \mathbf{R}^N . Suppose that $R(x) \geq \epsilon > 0$ on H . Then there is an integer d such that the polynomial $s^d R$ has all positive coefficients.*

We state a simple corollary of Theorem 3.1 or Theorem 3.2 (going back to Poincaré) that can be proved by high school mathematics. The result fails in the real-analytic or smooth settings. See [D3] and [HLP] for more information.

Corollary 3.2. *Let p be a polynomial in one real variable. Then $p(t) > 0$ for all $t \geq 0$ if and only if there is an integer d such that the polynomial given by $(1+t)^d p(t)$ has only positive coefficients. The minimum such d can be arbitrarily large for polynomials of fixed degree.*

See Section 9 for another circumstance where we gain insight into the general Hermitian case by considering real polynomials depending on only the variables $|z_1|^2, \dots, |z_n|^2$. We close this section by sketching the proof of Theorem 3.1.

Proof. The equivalence of items 2) and 3) follows from Lemma 3.2. Either implies item 4), which implies that r is positive away from the origin, and hence implies item 1). We discuss item 5) later. The crux of the matter is to prove that item 1) implies item 3).

We want to find an integer d such that $\langle z, w \rangle^d r(z, \bar{w})$ is the integral kernel of a positive operator. In order to place all these operators on the same footing, we study the operator $PM_{r(z, \bar{w})}$ with integral kernel equal to

$$B(z, \bar{w})r(z, \bar{w}) = \sum_{j=0}^{\infty} c_j \langle z, \bar{w} \rangle^j r(z, \bar{w}). \quad (27)$$

Recall that each c_j is a positive number. Let $\chi = \chi(w)$ be a non-negative smooth function which is positive at 0 and has compact support in B_n . Consider the operator $M_{r(z, \bar{z})}P + PM_{\chi}$ with integral kernel

$$(r(z, \bar{z}) + \chi(w))B(z, \bar{w}). \quad (28)$$

We add and subtract to obtain

$$\begin{aligned} B(z, \bar{w})r(z, \bar{w}) = \\ B(z, \bar{w})(r(z, \bar{w}) - r(z, \bar{z})) + B(z, \bar{w})(r(z, \bar{z}) + \chi(w)) - B(z, \bar{w})\chi(w). \end{aligned} \quad (29)$$

The three terms in (29) define the integral kernels of operators S_1 , S_2 , and S_3 . The operator S_3 is compact on all of $L^2(B_n)$. The operator S_2 is easily seen to be positive on $A^2(B_n)$. The operator S_1 can be written as

$$\sum_{a,b} c_{ab} M_{z^a} [P, M_{\bar{z}^b}]. \quad (30)$$

We claim that the operator in (30) is also compact; it is a finite sum of bounded operators times commutators of P with bounded operators. Such commutators are compact by Lemma 3.1. The composition of a bounded operator with a compact operator is compact, and a finite sum of compact operators is compact. Hence S_1 is compact. It follows that the operator $PM_{r(z, \bar{w})}$ is the sum of a compact operator and a positive operator. Hence, outside of a finite-dimensional subspace, this operator is itself positive. In other words, for d sufficiently large, $c_d \langle z, w \rangle^d r(z, \bar{w})$ is the kernel of a positive operator. Since $c_d > 0$, item 3) follows.

It remains only to check that item 5) is equivalent to the other statements. Assume that $r = \|P\|^2 - \|N\|^2$. Then we obtain

$$\|H_d\|^2 r = \|H_d \otimes P\|^2 - \|H_d \otimes N\|^2. \quad (31)$$

If 5.1) and 5.2) hold, then the right hand side of (31) becomes

$$\|\sqrt{I - L^*L}(H_d \otimes P)\|^2 = \|g\|^2. \quad (32)$$

If 5.3) also holds, then we obtain 4), and hence item 5) implies item 4). Conversely suppose that item 4) holds. Then the right-hand side of (31) is a squared norm $\|g\|^2$. We obtain

$$\|H_d \otimes P\|^2 = \|g\|^2 + \|H_d \otimes N\|^2 = \|g \oplus (H_d \otimes N)\|^2. \quad (33)$$

By [D1] there is a unitary map U such that

$$U((H_d \otimes P) \oplus 0) = (H_d \otimes N) \oplus g.$$

Letting L be one of the blocks of U gives 5.2), and 5.1) follows because U is unitary. The assumption that $\mathbf{V}(g) = \{0\}$ gives 5.3). \square

This decisive theorem has several useful consequences. We pause to prove one such result; others appear in the next two sections.

Theorem 3.3. *Suppose that $r(z, \bar{z})$ is a polynomial that is positive on the unit sphere S^{2n-1} . Then r agrees with the squared norm of a holomorphic polynomial mapping on S^{2n-1} .*

Proof. We sketch the proof. Let C be a positive number, to be chosen momentarily. Consider the function R_C defined by

$$R_C(z, t, \bar{z}, \bar{t}) = Hr(z, t, \bar{z}, \bar{t}) + C(\|z\|^2 - |t|^2)^m. \quad (34)$$

Here Hr , the bihomogenization of r , has total degree $2m$. We may assume without loss of generality that m itself is even. Note that R_C is bihomogeneous. Suppose we can choose C so that R_C is positive on the unit sphere. By Theorem 3.1 it follows that there is an integer d and a holomorphic polynomial mapping g such that

$$(\|z\|^2 + |t|^2)^d R_C(z, t, \bar{z}, \bar{t}) = \|g(z, t)\|^2. \quad (35)$$

Putting $t = 1$ and then $\|z\|^2 = 1$ gives

$$2^d r(z, \bar{z}) = \|g(z, 1)\|^2$$

on the sphere, and hence yields the conclusion of the Theorem.

The intuition is simple. It suffices to show that R_C is positive on $\|z\|^2 + |t|^2 = 2$. When $\|z\|^2 = |t|^2 = 1$, we know that R_C is positive, because r is positive on the sphere. By continuity, $R_C > 0$ when $|\|z\|^2 - |t|^2|$ is small. But, when this quantity is large (at most 2 of course), the second term in (34) is large and positive. Since the first term achieves a minimum on a compact set, we can choose C large enough to guarantee that $R_C > 0$ away from the origin. \square

The example $(|z_1|^2 - |z_2|^2)^2$ shows that non-negativity does not suffice for the conclusion.

Next we mention some related results concerning positive functions on the boundaries of strongly pseudoconvex domains. Løw [L] proved the following result. Suppose that Ω is a strongly pseudoconvex domain with C^2 boundary, and ϕ is a positive continuous function on the boundary $b\Omega$. Then there is a mapping g , holomorphic on Ω , continuous on the closure of Ω , and taking values in a finite dimensional space, such that $\phi = \|g\|^2$ on $b\Omega$. Lempert ([L1], [L2]) considers strongly pseudoconvex domains with real-analytic boundary. One of his results states that, given a positive continuous function ϕ on $b\Omega$, there is a sequence h_1, h_2, \dots of functions, holomorphic on Ω , continuous on $b\Omega$, such that $\|h\|^2 = \sum_j |h_j|^2$ converges on $b\Omega$ and agrees with ϕ there. These theorems form part of work concerning embedding strongly pseudoconvex domains into balls.

Given Løw's result, it is natural to ask whether Theorem 3.3 can be generalized. Recently Putinar and Scheiderer [PS] provided an important example and a new technique concerning such generalizations. The author once asked the following question, which, as Example shows 3.1 shows, has a negative answer in general. Let Ω be a strongly pseudoconvex domain with an algebraic boundary. Let $f(z, \bar{z})$ be a polynomial and assume that f is positive on $b\Omega$. Is there a holomorphic polynomial mapping g , taking values in a finite dimensional space, such that $f(z, \bar{z}) = \|g(z)\|^2$ on $b\Omega$. The answer can be no! The following example also shows that the holomorphic mapping g constructed by Løw does not extend holomorphically past the boundary, even when the data $b\Omega$ and ϕ are algebraic.

Example 3.1. Put $r(z, \bar{z}) = |z_1(z_1^2 - 1)|^2 + |z_2|^2 - c^2$. Let Ω be the set of z for which $r(z, \bar{z}) < 0$. Put $f(z, \bar{z}) = m - |z_1|^2|z_2|^2$. For sufficiently small positive c , Ω is strongly pseudoconvex. For M sufficiently large, $f > 0$ on $b\Omega$. But f agrees with no squared norm on $b\Omega$. The idea of the proof, due to Putinar and Scheiderer, amounts to considering the space $\mathcal{P}_2(b\Omega)$. Let $p = (1, c)$ and let $q = (-1, c)$. Simple calculation shows that

$$r(p, \bar{p}) = r(q, \bar{q}) = r(p, \bar{q}) = r(q, \bar{p}) = 0.$$

If $f = \|g\|^2$ on $b\Omega$, then we would have the following:

$$m - c^2 = f(p, \bar{p}) = \|g(p)\|^2$$

$$m - c^2 = f(q, \bar{q}) = \|g(q)\|^2$$

$$m + c^2 = f(p, \bar{q}) = f(q, \bar{p}) = \langle g(p), g(q) \rangle.$$

If these three conditions held, then the Cauchy-Schwarz inequality would imply the obviously false inequality

$$-4mc^2 = (m - c^2)^2 - (m + c^2)^2 \geq 0.$$

Dropping the term $|z_2|^2$ from the defining equation in Example 3.1 leads to an example of a domain in \mathbf{C} where the positivity property fails as well. The author believes that the original question should be rephrased along the following lines.

Let X be an algebraic subset of \mathbf{C}^n . One wishes to introduce the notation $\mathcal{P}_k(X)$ with the following meaning. Assume $X = \{u = 0\}$, and let z_1, \dots, z_k be points such that $u(z_j, \bar{z}_k) = 0$ for all j, k . When $j = k$ we see that $z_j \in X$; for $j \neq k$ we see that z_k is in the Segre set determined by z_j . A Hermitian polynomial f is in $\mathcal{P}_k(X)$ if each matrix $f(z_j, \bar{z}_k)$, formed by evaluation at such points, is nonnegative definite. In Example 3.1, we see that the given f is not in $\mathcal{P}_2(b\Omega)$. This approach leads to a subtle difficulty: the number k depends on the choice of defining equation u . For example, the unit sphere can be defined for each positive integer d by $u = \|z^{\otimes d}\|^2 - 1$. Each such function is a unit times $\|z\|^2 - 1$. After polarization, however, this property no longer holds. When $d = 1$, one cannot find distinct points z_1 and z_2 satisfying the above equations. For general d , however, one can find such sets with d distinct points. It therefore follows that one must define the appropriate notions for real polynomial ideals, rather than for their zero sets. Doing so leads to a notion of *Hermitian complexity* for real polynomial ideals, introduced in [DP].

It is also natural to expect that a stability result holds; appropriate information on the ideal tells us how large k needs to be. For example, by Theorem 3.3, for the ideal $\|z\|^2 - 1$ and for f strictly positive, we need only consider $k = 1$. For the ideal generated by r from Example 3.1, $k = 1$ does not suffice.

4. THE ONE-DIMENSIONAL CASE

We return to our analysis of \mathcal{Q} and \mathcal{Q}' . By Lemma 2.1 we gain information about these sets by pulling back to one dimension. Following but improving [D4] we completely analyze the one-dimensional case. Thus $n = 1$ in this section.

First we introduce the reflection of a Hermitian polynomial. This concept suggests that the Riemann sphere (rather than \mathbf{C}) is the right place to work.

Definition 4.1. Let $r(z, \bar{z})$ be a Hermitian symmetric polynomial of degree m in $z \in \mathbf{C}$. We define a new Hermitian symmetric polynomial r^* called the *reflection* of r by

$$r^*(z, \bar{z}) = |z|^{2m} r\left(\frac{1}{z}, \frac{1}{\bar{z}}\right).$$

Remark 4.1. The reflection is closely related to the bihomogenization:

$$r^*(z, \bar{z}) = (Hr)(1, z, 1, \bar{z}).$$

This formula requires that $n = 1$.

Definition 4.1 is a bit subtle. For example, the reflection map is not injective, and the reflection of a sum need not be the sum of the reflections. Reflection preserves neither degree in z nor total degree. Also, r^{**} need not be r .

Example 4.1. We compute three reflections:

- Put $r(z, \bar{z}) = 1 + (z + \bar{z})^4 + |z|^2$. Then $r^*(z, \bar{z}) = |z|^8 + (z + \bar{z})^4 + |z|^6$.
- Put $r(z, \bar{z}) = z^m + \bar{z}^m$. Then $r^* = r$.
- Put $r(z, \bar{z}) = |z|^{2k}$. Then, for all k , $r^*(z, \bar{z}) = 1$.

Example 4.2. If $r(z, \bar{z}) = z^2 + \bar{z}^2$ and $s(z, \bar{z}) = z^3 + \bar{z}^3$, then each is its own reflection by the previous example. But

$$(r+s)^*(z, \bar{z}) = |z|^6 \left(\frac{1}{z^2} + \frac{1}{\bar{z}^2} + \frac{1}{z^3} + \frac{1}{\bar{z}^3} \right) = |z|^2(z^2 + \bar{z}^2) + z^3 + \bar{z}^3 \neq r(z, \bar{z})^* + s(z, \bar{z})^*.$$

On the other hand we have the following useful statement, which we apply in the proof of Theorem 4.1. We will also apply the subsequent lemma and its corollary

Lemma 4.1. $r \in \mathcal{Q}$ if and only if $r^* \in \mathcal{Q}$. Also, $r \in \mathcal{Q}'$ if and only if $r^* \in \mathcal{Q}'$.

Proof. By the symmetry between 0 and infinity, it suffices to prove one implication in each case. Suppose $r \in \mathcal{Q}$. Then $Hr \in \mathcal{Q}$. Remark 4.1 implies that $r^* \in \mathcal{Q}$. The proof for \mathcal{Q}' is essentially the same. \square

Lemma 4.2. Let r be a Hermitian symmetric polynomial in one variable. Assume $r(p, \bar{p}) = 0$. If $r \in \mathcal{Q}'$, then r is divisible (as a polynomial) by $|z - p|^2$.

Proof. If $sr = \|f\|^2$, then the hypothesis implies $\|f(p)\|^2 = 0$. We see that $f_j(p) = 0$ for all j , and hence each component f_j is divisible by $z - p$. We may cancel all factors of $|z - p|^2$ that divide s from both sides of the equation. Since r is Hermitian and $r(p, \bar{p}) = 0$, both $(z - p)$ and its conjugate divide r . Thus r is divisible by $|z - p|^2$. \square

Corollary 4.1. Assume $r \in \mathcal{Q}'$ and r contains pure terms (z^k or \bar{z}^k). Then $r(0, 0) > 0$.

Proof. If $r(0, 0) = 0$, then the lemma implies r is divisible by $|z|^2$ and hence has no pure terms. \square

Example 4.3. Put $r(z, \bar{z}) = |z|^2 + (z + \bar{z})^4 + |z|^6$. Then r is not in \mathcal{Q}' .

In Example 4.3, r has an isolated 0 at 0, but the z^4 term prevents r from being in \mathcal{Q}' . The problem is that the zero-set of r is not properly defined.

Even in one dimension we must deal with the following point. There exist Hermitian symmetric polynomials whose values are (strictly) positive, yet for which

the infimum of the set of values is zero. Such polynomials cannot be quotients of squared norms.

Example 4.4. Put $f(x, y) = (xy - 1)^2 + x^2$. For $x > 0$, we have $f(x, \frac{1}{x}) = x^2$ and hence f achieves values arbitrarily close to 0. On the other hand $f(x, y)$ is evidently never 0. Writing f in terms of z and \bar{z} gives a Hermitian symmetric example.

We next state and prove Theorem 4.1. It is particularly striking that the sets \mathcal{Q} and \mathcal{Q}' are the same. These sets are characterized by a simple condition on degree, which provides the extra thing needed besides a properly defined zero-set.

Theorem 4.1. *Let $r \in \mathcal{P}_1(1)$ be a Hermitian symmetric non-negative polynomial in one variable. The following are equivalent:*

- 1) *There is a holomorphic polynomial h such that $r = |h|^2 R$, $R > 0$, and the total degree of R is twice the degree of R in z .*
- 2) *There is a holomorphic polynomial h such that $r = |h|^2 R$ and $R \in \mathcal{Q}$.*
- 3) *$r \in \mathcal{Q}$.*
- 4) *There is a holomorphic polynomial h such that $r = |h|^2 R$ and $R \in \mathcal{Q}'$.*
- 5) *$r \in \mathcal{Q}'$.*

Proof. First suppose that r vanishes identically. If we take $h = 0$ and $R = 1$, then all the statements hold. Henceforth we assume that r does not vanish identically. Suppose 1) holds. First we note by the information on degree that there is a unique term $c|z|^{2m}$ of highest degree, where $c > 0$. Set $\epsilon = \inf(R)$. We claim that $\epsilon > 0$. Assuming this claim, consider the bihomogenization HR . For $t \neq 0$ we have

$$HR(z, t, \bar{z}, \bar{t}) = |t|^{2m} R\left(\frac{z}{t}, \frac{\bar{z}}{\bar{t}}\right) \geq |t|^{2m} \epsilon.$$

For t near 0 however the values of HR are near $c|z|^{2m}$. Hence, there is a positive constant δ such that

$$HR(z, t, \bar{z}, \bar{t}) \geq \delta(|z|^{2m} + |t|^{2m}).$$

Therefore HR is strictly positive away from the origin in \mathbf{C}^{n+1} . By Theorem 3.1, we conclude that $HR \in \mathcal{Q}(2)$. We recover R from HR by setting $t = 1$. Hence $R \in \mathcal{Q}(1)$. We obtain r by multiplication by $|h|^2$; by Lemma 2.4, $r \in \mathcal{Q}(1)$. Since $\mathcal{Q} \subset \mathcal{Q}'$, we obtain 4) from 2) and 5) from 3). Thus, given the claim, 1) implies 2) implies 3) implies 5) and 2) implies 4).

It remains to prove the claim. By the assumption on degree of R , there is a unique term $c|z|^{2m}$ of highest degree. Hence, for $|z|$ sufficiently large, we have

$$R(z, \bar{z}) \geq \frac{c}{2}|z|^{2m}. \quad (36)$$

Now suppose that $\inf(R) = 0$. We can then find a sequence z_ν on which $R(z_\nu, \bar{z}_\nu)$ tends to zero. Since R is bounded below by a positive number on any compact set, we may assume that $|z_\nu|$ tends to infinity. But setting $z = z_\nu$ violates (36). We have now shown that 1) implies the rest of the statements. We finish by showing that 5) implies 1).

Assume that 5) holds; thus there is an s for which $rs = \|f\|^2$. As above, if $r = 0$ all the statements hold. Otherwise $\mathbf{V}(r)$ must be the zero-set of a holomorphic polynomial h , which we may assume is $h(z) = \prod(z - p_j)$. Both sides of $rs = \|f\|^2$ are divisible by $|h|^2$. We put $R = \frac{r}{|h|^2}$ and we see that $R \in \mathcal{Q}'$. Furthermore $R > 0$. It remains to establish the information about its degree. Suppose that the terms of

degree $2m$ include a term of degree larger than m in z . It follows that the reflected polynomial R^* vanishes at the origin and yet contains pure terms. By Corollary 4.1, R^* is not in \mathcal{Q}' , and by Lemma 4.1 R is not in \mathcal{Q}' . Hence no such term can exist. Thus 5) implies 1). Hence all the statements are equivalent. \square

Corollary 4.2. $\mathcal{Q}(1) = \mathcal{Q}'(1)$.

Corollary 4.3. *Let $r(z, \bar{z})$ be a Hermitian symmetric polynomial in one variable. Then $r \in \mathcal{Q}$ if and only if the following holds:*

Either r vanishes identically, or the zero-set of r is a finite set $\{p_1, \dots, p_K\}$ (repetitions allowed) such that

$$r(z, \bar{z}) = \prod_{j=1}^K |z - p_j|^2 s(z, \bar{z}),$$

and s satisfies both of the following conditions:

- 1) s is strictly positive.*
- 2) The total degree of s is twice the degree of s in z .*

Corollary 4.4. *Suppose $r > 0$ but $\inf(r) = 0$. Then r is not in \mathcal{Q}' .*

Proof. This statement is a corollary of the proof of Theorem 4.1. \square

The general one-dimensional case relies on the (non-degenerate) bihomogeneous case in two dimensions. After dividing out factors of the form $|h(z)|^2$, we reduce to the situation where hr satisfies the hypotheses of Theorem 3.1.

Consider any Hermitian symmetric polynomial with pure terms $2(z^k + \bar{z}^k)$. We may write these terms as

$$2(z^k + \bar{z}^k) = |z^k + 1|^2 - |z^k - 1|^2 = |f_0|^2 - |g_0|^2.$$

Adding any $\|f'\|^2 - \|g'\|^2$ to the right hand side, where $f(0) = 0$ and $g(0) = 0$, yields a function r of the form

$$r = \|f'\|^2 + |f_0|^2 - \|g'\|^2 - |g_0|^2 = \|f\|^2 - \|g\|^2. \quad (37)$$

Putting $z = 0$ in (37) shows that there is no constant c such that $c < 1$ and $\|g\|^2 \leq c\|f\|^2$. Hence the failure of such a constant to exist eliminates pure terms. In fact, the existence of such a constant gives Varolin's characterization (Theorem 5.1) of $\mathcal{Q}(n)$ in all dimensions.

Pulling back to one dimension. Assume we are in n dimensions, where $n \geq 2$. We can combine Theorem 4.1 and Lemma 2.1 to give easily checkable necessary conditions for being in \mathcal{Q} or \mathcal{Q}' . First we note the following simple fact.

Lemma 4.3. *Suppose $r \in \mathcal{Q}$. Then $\mathbf{V}(r)$ is a complex algebraic variety. Suppose $r \in \mathcal{Q}'$ and r is not identically zero. Then $\mathbf{V}(r)$ is contained in a complex algebraic variety of positive codimension.*

Definition 4.2. A Hermitian symmetric polynomial $r : \mathbf{C} \rightarrow \mathbf{R}$ satisfies property (W) if either r is identically 0, or r vanishes to finite even order $2m$ at 0 and its initial form (terms of lowest total degree) is $c|t|^{2m}$.

In other words, the only term of lowest total degree is $c|t|^{2m}$. For example, $(2\operatorname{Re}(t))^2$ does not satisfy property (W). It equals its initial form, which is $t^2 + 2|t|^2 + \bar{t}^2$.

The following lemma from [D3] shows that property (W) is necessary for being a quotient of squared norms. Note also its relationship with Lemma 4.2 when $p = 0$.

Lemma 4.4. *Suppose that $r \in \mathcal{Q}$. Let $t \rightarrow z(t)$ be a polynomial mapping. Then the pullback function $t \rightarrow r(z(t), \overline{z(t)})$ satisfies property (W).*

Example 4.5. Non-negative bihomogeneous polynomials not in \mathcal{Q}' :

$$r(z, \bar{z}) = (|z_1|^2 - |z_2|^2)^2 \quad (38)$$

$$h(z, \bar{z}) = (|z_1 z_2|^2 - |z_3|^4)^2 + |z_1|^8 \quad (39)$$

The zero-set of the polynomial r from (38) is three real dimensions, and hence not contained in any complex variety other than the whole space. Thus r is not in \mathcal{Q}' and hence not in \mathcal{Q} either. Alternatively, property (W) fails if we pullback using $z(t) = (1 + t, 1)$, obtaining

$$p(z(t), \overline{z(t)}) = (t + \bar{t} + |t|^2)^2.$$

This expression also violates the condition of Lemma 4.2.

The zero-set of h is the complex variety defined by $z_1 = z_3 = 0$. Yet h is not in \mathcal{Q} . Put $z(t) = (t^2, 1 + t, t)$. Then property (W) fails for the pullback. A simple computation shows that the initial form of the pullback contains the term $t^4 \bar{t}^6$.

These examples prove that the containment $\mathcal{Q} \subset \mathcal{P}_1$ is strict.

5. VAROLIN'S THEOREMS

Varolin has extended Theorem 3.1 in two fundamental ways. The first way to extend the result is to allow objects more general than polynomials. We may regard a homogeneous polynomial of degree m on \mathbf{C}^n as a section of the m -th power \mathcal{H}^m of the hyperplane bundle over complex projective space \mathbf{P}_{n-1} . Hence a Hermitian symmetric polynomial

$$\sum c_{\alpha\beta} z^\alpha \bar{z}^\beta$$

can be rewritten

$$\sum c_{\alpha\beta} s_\alpha \bar{s}_\beta, \quad (40)$$

where the s_α form a basis for sections of \mathcal{H}^m . If such a polynomial is non-negative, then it can be regarded as a metric on the dual line bundle. Many of the ideas of this paper extend to metrics on holomorphic line bundles over compact complex manifolds. We briefly discuss some of this material in Section 8.

Our main purpose in this current section is the other direction in which Varolin extended Theorem 3.1. Suppose that r is non-negative and bihomogeneous, but vanishes outside the origin. We have seen that r is sometimes in \mathcal{Q} but other times it is not. Varolin proved the following two results, by generalizing the proof of Theorem 3.1 and using a form of the resolution of singularities, giving a complete solution to Analogue 1. The version of Theorem 5.1 differs in language from Theorem 1 as stated in [V], but the two statements are easily seen to be equivalent. The version of Theorem 5.2 is essentially Proposition 4.2 in [V].

Theorem 5.1 (Varolin). *Suppose $r = \|f\|^2 - \|g\|^2$ is a bihomogeneous polynomial and the components of f and g are linearly independent. Then $r \in \mathcal{Q}$ if and only if there is a $\lambda < 1$ such that $\|g\|^2 \leq \lambda \|f\|^2$.*

Theorem 5.2. *Suppose r is as in Theorem 5.1. Then $r \in \mathcal{Q}$ if and only if Property (W) holds for z^*r for every rational map $z : \mathbb{C} \rightarrow \mathbb{C}^n$.*

By combining Theorem 5.2 with Theorem 4.1, we obtain a complete solution to Analogue 2.

Theorem 5.3. *For all n , $\mathcal{Q}(n) = \mathcal{Q}'(n)$. In other words, let r be a Hermitian symmetric polynomial, not identically 0. Then r is a quotient of squared norms if and only if it divides a squared norm.*

Proof. The containment $\mathcal{Q} \subset \mathcal{Q}'$ is trivial. Suppose that r is not in $\mathcal{Q}(n)$. Then the bihomogenization Hr is not in $\mathcal{Q}(n+1)$. By Theorem 5.2 there is a rational curve z for which $z^*(Hr)$ violates property W. After clearing denominators it follows that $z^*(Hr)$ is not in $\mathcal{Q}(1)$. By Theorem 4.1, $z^*(Hr)$ is not in $\mathcal{Q}'(1)$ either. But then Hr is not in $\mathcal{Q}'(n+1)$ and hence r is not in $\mathcal{Q}'(n)$. Hence $\mathcal{Q}(n) = \mathcal{Q}'(n)$. \square

It is possible to prove Theorem 5.3 by using Varolin's approach via the resolution of singularities. The idea, roughly speaking, follows. Assume $rs = \|f\|^2$. Blow up the ideal of $\|f\|^2$ and cancel factors to reduce to the case where r and s are positive. By Theorem 3.1 each is an element of \mathcal{Q} . The proof given here is similar in spirit. Theorem 5.2 of Varolin enables the reduction to the one-dimensional case. Theorem 4.1 of this paper handles the one-dimensional case, although the logic still passes through Theorem 3.1.

6. APPLICATIONS TO PROPER MAPPINGS BETWEEN BALLS

We first recall some facts about proper mappings between domains in complex Euclidean spaces. Let Ω and Ω' be bounded domains in \mathbb{C}^n and \mathbb{C}^N . A holomorphic mapping $f : \Omega \rightarrow \Omega'$ is proper if $f^{-1}(K)$ is compact in Ω whenever K is compact in Ω' . When such an f extends to be a continuous mapping of the boundaries, it will be proper precisely when it maps the boundary $b\Omega$ to the boundary $b\Omega'$.

Let $f : B_n \rightarrow B_N$ be a proper holomorphic mapping. When f extends smoothly to the boundary we see that $\|z\|^2 = 1$ implies $\|f(z)\|^2 = 1$, and hence there is an obvious connection to squared norms. We will see more subtle relationships as well.

We recall many facts about proper holomorphic mappings $f : B_n \rightarrow B_N$. See [F] and [D2] for references.

- When $N < n$, every such f is a constant. This conclusion follows from the observation that positive dimensional complex analytic subvarieties of the ball are noncompact.
- When $n = N = 1$, every such f is a finite Blaschke product. There are finitely many points a_j in B_1 , positive integer multiplicities m_j , and an element $e^{i\theta}$ such that

$$f(z) = e^{i\theta} \prod_{j=1}^K \left(\frac{z - a_j}{1 - \bar{a}_j z} \right)^{m_j} \quad (41)$$

Note that (41) shows that there is no restriction on the denominator. Every polynomial q that is not zero on the closed ball arises as the denominator of a rational function reduced to lowest terms.

- When $n = N > 1$, a proper holomorphic map $f : B_n \rightarrow B_n$ is necessarily an automorphism. In particular f is a linear fractional transformation with denominator $1 - \langle z, a \rangle$ for $a \in B_n$.
- There are proper holomorphic mappings $f : B_n \rightarrow B_{n+1}$ that are continuous but not smooth on the boundary sphere.
- Assume $n \geq 2$. If $f : B_n \rightarrow B_N$ is a proper map and has $N - n + 1$ continuous derivatives on the boundary, then f must be a rational mapping [F]. Furthermore, by [CS], the denominator cannot vanish on the closed ball.

The author began his study of complex variables analogues of Hilbert's problem in order to verify the next result. In it we want $\frac{p}{q}$ to be in lowest terms, or else we have the trivial example where $p(z) = q(z)(z_1, \dots, z_n)$.

Theorem 6.1. *Let $q : \mathbf{C}^n \rightarrow \mathbf{C}$ be a holomorphic polynomial, and suppose that q does not vanish on the closed unit ball. Then there is an integer N and a holomorphic polynomial mapping $p : \mathbf{C}^n \rightarrow \mathbf{C}^N$ such that*

1. $\frac{p}{q}$ is a rational proper mapping between B_n and B_N .
2. $\frac{p}{q}$ is reduced to lowest terms.

Proof. The result is trivial when q is a constant and it is easy when $n = 1$. When the degree d of q is positive in one dimension, we define p by $p(z) = z^d \overline{q}(\frac{1}{z})$. Such a proof cannot work in higher dimensions. The minimum integer N can be arbitrarily large even when $n = 2$ and the degree of q is also two.

Now assume $n \geq 2$. Suppose that $q(z) \neq 0$ on the closed ball. Let g be an arbitrary polynomial such that q and g have no common factor. Then there is a constant c so that

$$|q(z)|^2 - |cg(z)|^2 > 0 \quad (42)$$

for $\|z\|^2 = 1$. We set $p_1 = cg$.

By Theorem 3.3, $|q|^2 - |p_1|^2$ agrees on the sphere with a squared norm of a holomorphic polynomial mapping. Thus there are polynomials p_2, \dots, p_N such that

$$|q(z)|^2 - |p_1(z)|^2 = \sum_{j=2}^N |p_j(z)|^2 \quad (43)$$

on the sphere. It then follows that $\frac{p}{q}$ does the job. \square

Theorem 3.3 can be used also to show that one can choose various components p of a proper holomorphic polynomial mapping arbitrarily, assuming only that they satisfy the necessary condition $\|p(z)\|^2 < 1$ on the sphere.

Corollary 6.1. *Let $p : \mathbf{C}^n \rightarrow \mathbf{C}^k$ be a polynomial with $\|p(z)\|^2 < 1$ on the unit sphere. Then there is a polynomial mapping g such that the polynomial $p \oplus g$ is a proper holomorphic mapping between balls.*

Proof. Note that $1 - \|p(z)\|^2$ is a polynomial that is positive on the sphere. Hence we can find a holomorphic polynomial mapping g such that

$$1 - \|p(z)\|^2 = \|g(z)\|^2$$

on the sphere. We may assume that not both p and g are constant. Then $p \oplus g$ is a non-constant holomorphic polynomial mapping whose squared norm equals unity on the sphere. By the maximum principle $p \oplus g$ is the required mapping. \square

7. ROOTS OF SQUARED NORMS

We check a simple fact noted in the introduction. If $r \in \text{rad}(\mathcal{P}_\infty)$, then we have $r^{N-1}r \in \mathcal{P}_\infty$. Thus there exists a q for which $qr \in \mathcal{P}_\infty$. Hence $r \in \mathcal{Q}'$. Thus

$$\text{rad}(\mathcal{P}_\infty) \subset \mathcal{Q}'.$$

In this section we provide additional information about $\text{rad}(\mathcal{P}_\infty)$. Perhaps the most striking statement is its relationship with \mathcal{P}_2 . We may regard \mathcal{P}_2 as a closed cone in real Euclidean space. We have the following result, in which int denotes interior.

Theorem 7.1. *The following containments hold, and all are strict:*

$$\text{int}(\mathcal{P}_2) \subset \text{rad}(\mathcal{P}_\infty) \subset \mathcal{P}_2 \subset \mathcal{L} \subset \mathcal{P}_1. \quad (44)$$

Corollary 7.1. *Suppose $r \in \text{rad}(\mathcal{P}_\infty)$. Then $r \in \mathcal{P}_2$ and $\log(r)$ is plurisubharmonic.*

We discuss but do not prove the first containment. We begin with a surprising fact and continue by establishing the other containments.

Example 7.1. $\text{rad}(\mathcal{P}_\infty)$ is not closed under sum. Choose an $R \in \text{rad}(\mathcal{P}_\infty)$ of the form $r = \|f\|^2 - |g|^2$, where f and g are homogeneous of degree m in the variables z_2 and z_3 and their components are linearly independent. Let $r = |z_1|^{2m}$. Then $R + r$ is not in $\text{rad}(\mathcal{P}_\infty)$; for each N the function $(R + r)^N$ will contain the term $-N|g|^2|z_1|^{2m(N-1)}$. This term arises nowhere else in the expansion, and hence $(R + r)^N$ is not a squared norm. If we can find such an R in $\text{rad}(\mathcal{P}_\infty)$, we have an example. By Example 2.1, $R = (|z_2|^2 + |z_3|^2)^4 - \beta|z_2z_3|^4$ works if $6 < \beta < 8$.

The function $R = R_\beta$ from this example shows that $\text{rad}(\mathcal{P}_\infty)$ is not closed under limits. It is easy to see that $\text{rad}(\mathcal{P}_\infty)$ is closed under products. We next establish most of the containments from Theorem 7.1.

Lemma 7.1. $\text{rad}(\mathcal{P}_\infty) \subset \mathcal{P}_2$.

Proof. Suppose that $r^N = \|f\|^2$. By the usual Cauchy-Schwarz inequality,

$$(r(z, \bar{z})r(w, \bar{w}))^N = \|f(z)\|^2\|f(w)\|^2 \geq |\langle f(z), f(w) \rangle|^2 = |r(z, \bar{w})|^{2N}. \quad (45)$$

Since $N > 0$, we may take N -th roots of both sides of (45) and preserve the direction of the inequality to obtain (13). By the principal minors test for non-negative definiteness, we see that $r \in \mathcal{P}_2$. \square

Remark 7.1. Suppose r satisfies (13). If $r(z, \bar{z}) > 0$ for a single point z , then $r \in \mathcal{P}_2 \subset \mathcal{P}_1$. Positivity at one point is required. When r is minus a squared norm, and not identically zero, (13) holds and r is not in \mathcal{P}_1 . If $r(z, \bar{z})$ is positive at one point, then (13) is equivalent to

$$r(w, \bar{w}) \geq \frac{|r(z, \bar{w})|^2}{r(z, \bar{z})}$$

and hence $r(w, \bar{w})$ is non-negative for all w .

Lemma 7.2. $\mathcal{P}_2 \subset \mathcal{L}$.

Proof. If $r \in \mathcal{P}_2$, then (13) holds. Since equality holds when $z = w$, the right-hand side of (13) has a minimum at $z = w$, and hence its complex Hessian there is non-negative definite. Computing the Hessian shows that the matrix with i, j entry equal to

$$r r_{z_i \bar{z}_j} - r_{z_i} r_{\bar{z}_j}$$

is non-negative definite. Computing the Hessian of $\log(r)$ leads to the same condition. \square

We continue to develop a feeling for the Cauchy-Schwarz inequality. By (3) there are holomorphic polynomial mappings f and g , taking values in finite-dimensional spaces, such that

$$r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2.$$

We may assume that there are no linear dependence relations among the components of f and g , but even then the representation is not unique.

In the next Proposition we allow f and g to be Hilbert space valued holomorphic mappings. In the polynomial case the Hilbert space is finite-dimensional.

Proposition 7.1. *Suppose that \mathcal{H} is a Hilbert space, and that f and g are holomorphic mappings to \mathcal{H} . Put*

$$r(z, \bar{w}) = \langle f(z), f(w) \rangle - \langle g(z), g(w) \rangle \quad (46)$$

Then (13) holds if and only if, for every pair of points z and w , we have

$$\begin{aligned} & \|f(z) \otimes g(w) - f(w) \otimes g(z)\|^2 \\ & \leq \|f(z)\|^2 \|f(w)\|^2 - |\langle f(z), f(w) \rangle|^2 + \|g(z)\|^2 \|g(w)\|^2 - |\langle g(z), g(w) \rangle|^2 \end{aligned} \quad (47)$$

Proof. Begin by using (46) to express (13) in terms of f and g . The resulting inequality is then seen to be equivalent to (47). To see this, expand the squared norm on the left side of (47), and use the identity $\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle = \langle u_1, u_2 \rangle \langle v_1, v_2 \rangle$. \square

Consider the right side of (47); the two terms involving f , and the two terms involving g , are each non-negative by the usual Cauchy-Schwarz inequality. Their sum is thus non-negative. Version (47) of the Cauchy-Schwarz inequality demands more; their sum must bound an obviously non-negative expression that reveals the symmetry of the situation. The left side of (47) has the interpretation as $\|(f \wedge g)(z, w)\|^2$, but the author does not know how to use this information.

It remains to discuss the first containment in (44). It follows from results in [CR3] and [V]. These results, which are expressed in terms of metrics on line bundles, involve strict forms of (13). They imply, when r is bihomogeneous and satisfies a strict form of (13), that $R \in \text{rad}(\mathcal{P}_\infty)$. The strict forms of (13) are open conditions on the coefficients, and hence we obtain the first containment in (44).

8. ISOMETRIC IMBEDDING FOR HOLOMORPHIC BUNDLES

Let r be a bihomogeneous polynomial that is positive away from the origin in $\mathbf{C}^{\mathbf{N}+1}$. The link to bundles arises by first considering complex projective space $\mathbf{P}_{\mathbf{N}}$, the collection of lines through the origin in $\mathbf{C}^{\mathbf{N}+1}$. We have the usual open covering given by open sets U_j where $z_j \neq 0$. In U_j we define f_j by

$$f_j(z, \bar{z}) = \frac{r(z, \bar{z})}{|z_j|^{2m}}. \quad (48)$$

On the overlap $U_j \cap U_k$ these functions then transform via

$$f_k = \left(\frac{z_j}{z_k}\right)^m |f_j|. \quad (49)$$

Since $(\frac{z_j}{z_k})^m$ are the transition functions for the m -th power of the universal line bundle \mathbf{U}^m , the functions f_j determine a Hermitian metric on \mathbf{U}^m .

We will reformulate Theorem 3.1 in this language and then generalize it.

Let r be a bihomogeneous polynomial of degree $2m$. It defines via (48) a metric on \mathbf{U}^m if and only if it is positive as a function away from the origin. This metric is already a pullback of the Euclidean metric if and only if $r \in \mathcal{Q}$. Some tensor power of the bundle with itself is a pullback if and only if $r \in \text{rad}(\mathcal{P}_{\infty})$. If $r \in \mathcal{P}_2$, then $r \in \mathcal{L}$. This condition is equivalent to the negativity of the curvature of the bundle, or to the pseudoconvexity of the unit ball in the total space of the bundle.

The previous paragraph applies in particular to the function r_{λ} from Example 2.1. When $\lambda < 16$, r_{λ} is strictly positive away from the origin, and hence defines a metric on \mathbf{U}^4 over \mathbf{P}_1 . By varying the parameter λ we see that the various positivity properties of bundle metrics are also distinct.

We next restate Theorem 3.1.

Theorem 8.1. *Let (\mathbf{U}^m, r) denote the m -th power of the universal line bundle over $\mathbf{P}_{\mathbf{n}}$ with metric defined by r . Then there are integers N and d such that $(\mathbf{U}^{m+d}, ||z||^{2d}r(z, \bar{z}))$ is a (holomorphic) pullback $g^*(\mathbf{U}, ||L(\zeta)||^2)$ of the standard metric on the universal bundle over $\mathbf{P}_{\mathbf{N}}$. The mapping $g : \mathbf{P}_{\mathbf{n}} \rightarrow \mathbf{P}_{\mathbf{N}}$ is a holomorphic (polynomial) embedding and L is an invertible linear mapping.*

$$(\mathbf{U}^m, r) \otimes (\mathbf{U}^d, ||z||^{2d}) = (\mathbf{U}^{m+d}, ||z||^{2d}r(z, \bar{z})) = (\mathbf{U}^{m+d}, ||g(z)||^2)$$

We have the bundles and metrics

$$\begin{aligned} \pi_1 : (\mathbf{U}^m, r) &\rightarrow \mathbf{P}_{\mathbf{n}} \\ \pi_2 : (\mathbf{U}^{m+d}, ||z||^{2d}r) &\rightarrow \mathbf{P}_{\mathbf{n}} \\ \pi_3 : (\mathbf{U}, ||L(\zeta)||^2) &\rightarrow \mathbf{P}_{\mathbf{N}} \end{aligned}$$

Thus π_1 is not an isometric pullback of π_3 , but, for sufficiently large d , π_2 is such a pullback.

This formulation suggests a generalization to more general Hermitian bundles. See [CD3] for the precise definitions of globalizable metric and the needed sharp form of inequality (13). See [V] for an improved exposition that allows for degenerate metrics. A version of (13) arises also in Calabi's work [Cal] on isometric imbeddings of the tangent bundle. The main result of [CD3] is the following isometric imbedding theorem for holomorphic bundles.

Theorem 8.2. *Let M be a compact complex manifold. Let E be a vector bundle of rank p over M with globalizable Hermitian metric G . Let L be a line bundle over M with globalizable Hermitian metric R , and suppose that L is negative and that R satisfies a sharp form of (13). Then there is an integer d_0 such that, for all d with $d \geq d_0$, there is a holomorphic imbedding h_d with $h_d : M \rightarrow \mathbf{G}_{p,N}$ such that $E \otimes L^d = h_d^* \mathbf{U}_{p,N}$, and $GR^d = h_d^*(g_0)$.*

The special case where the base manifold is complex projective space \mathbf{P}_{n-1} gives us Theorem 3.3. We let E be a power \mathbf{U}^m of the universal bundle, with metric determined by the bihomogeneous polynomial r , and we let L be the universal bundle \mathbf{U} with the Euclidean metric. A matrix analogue of Theorem 3.1 holds, where E is the bundle of rank k given by k copies of \mathbf{U}^m . See [D3] for details.

Corollary 8.1. *Let $M(z, \bar{z})$ be a matrix of bihomogeneous polynomials of the same degree that is positive-definite away from the origin. Suppose R is a bihomogeneous polynomial that is positive away from the origin and satisfies a sharp form of (13). Assume also that $\{R < 1\}$ is a strongly pseudoconvex domain. Then there is an integer d and a matrix A of holomorphic homogeneous polynomials such that*

$$R(z, \bar{z})^d M(z, \bar{z}) = A(z)^* A(z). \quad (50)$$

In particular we can choose $R(z, \bar{z}) = \|z\|^{2d}$.

A matrix C is positive definite if and only if its components c_{ij} can be expressed as inner products $\langle e_i, e_j \rangle$ of basis elements; we can thus factor a positive definite matrix of constants as $C = A^* A$. When the entries depend real-analytically on parameters it is generally impossible to make A depend holomorphically on these parameters. Writing an operator-valued real-analytic function as in the right side of (50) is called *holomorphic factorization*. See [RR] for classical results about holomorphic factorization of operator-valued holomorphic functions of one complex variable. In the situation of Corollary 9.1, one cannot factor M holomorphically, but one can factor $R^d M$ holomorphically when d is sufficiently large.

9. SIGNATURE PAIRS

Let r be a Hermitian symmetric polynomial with $\mathbf{s}(r) = (A, B)$. The underlying matrix of coefficients of r is diagonal if and only if we can write

$$r = \sum_{\alpha} c_{\alpha} |z|^{2\alpha} = \sum_{\alpha} c_{\alpha} |z_1|^{2\alpha_1} |z_2|^{2\alpha_2} \dots |z_n|^{2\alpha_n}.$$

Define the moment map \mathbf{m} by

$$z \rightarrow x = (x_1, \dots, x_n) = (|z_1|^2, \dots, |z_n|^2) = \mathbf{m}(z). \quad (51)$$

Thus in the diagonal case there is a (real) polynomial R in x such that

$$r(z, \bar{z}) = R(\mathbf{m}(z)) = R(x). \quad (52)$$

Then R has A positive coefficients and B negative coefficients.

The relationship between the diagonal case and the general case parallels the relationship between Theorem 3.2 and Theorem 3.1. We show next that the special diagonal situation suffices for finding examples of maximal collapsing of rank.

Lemma 9.1. *Assume $m \geq 2$. For $t \in \mathbf{R}$ put $p(t) = t^{2^m} + 1$. Then there is a polynomial $q(t)$ such that*

- All $2^{m-1} + 1$ coefficients of q are positive.

- $p(t) = q(t)q(-t)$

Proof. Regard t as a complex variable. Put $\omega = e^{\frac{\pi i}{2^m}}$. The roots of p occur when t is a 2^m -th root of -1 , and hence are odd powers of ω . Factor p into linear factors:

$$p(t) = \prod_j (t - \omega^{2j+1}).$$

The roots are symmetrically located in the four quadrants. We define q by taking the product over the terms where $\operatorname{Re}(\omega^{2j+1}) < 0$. Each such factor has a corresponding conjugate factor. Hence

$$q(t) = \prod (t^2 - 2\operatorname{Re}(\omega^{2j+1})t + 1),$$

and all the coefficients of q are positive. The remaining terms in the factorization of p define $q(-t)$, and the result follows. \square

We illustrate Lemma 9.1 with an example. Set $a = \sqrt{4 \pm 2\sqrt{2}}$. Then

$$t^8 + 1 = (t^4 + at^3 + \frac{a^2}{2}t^2 + at + 1)(t^4 - at^3 + \frac{a^2}{2}t^2 - at + 1). \quad (53)$$

Bihomogenization leads to the following result from [DL].

Proposition 9.1. *There are Hermitian symmetric polynomials q and r such that the following hold:*

- q and r are each bihomogeneous of total degree 2^m .
- $\mathbf{s}(q) = (2^{m-1} + 1, 0)$.
- $\mathbf{s}(r) = (2^{m-2} + 1, 2^{m-2})$.
- $\mathbf{s}(qr) = (2, 0)$.

Corollary 9.1. *For each integer k of the form $2^{m-1} + 1$, there exists $r \in \mathcal{Q}$ of rank k such that $\|g\|^2 r = \|f\|^2$ and $\|f\|^2$ has rank 2.*

Consider this proposition and corollary from the point of view of starting with r . It is a non-negative Hermitian polynomial with signature pair $(2^{m-2} + 1, 2^{m-2})$ and rank $2^{m-1} + 1$. By Proposition 9.1, r is a quotient of squared norms, where the rank of the numerator is 2. For example, (53) provides an example of a polynomial $r \in \mathcal{Q}$ whose signature pair of r is $(3, 2)$. The rank of the numerator is 2 and the rank of the denominator is 5. The point of the Corollary is that by choosing larger values of m , we can make $\mathbf{r}(qr) = 2$, while the ranks of the factors are arbitrarily large. This phenomenon illustrates the same warning required in our discussion near (9.1) of Pfister's theorem in the real case.

The next result shows that we cannot decrease the rank to 1. On the other hand, its conclusion is false for real-analytic Hermitian symmetric functions. Consider the identity $1 = e^{\|z\|^2} e^{-\|z\|^2}$. If we expand the exponential as a series, then the signature pairs of the factors would be $(\infty, 0)$ and (∞, ∞) . Yet their product has signature pair $(1, 0)$. We return to the polynomial case.

Proposition 9.2. ([DL]) *Let p and q be Hermitian symmetric polynomials with $\mathbf{r}(pq) = 1$. Then $\mathbf{r}(p) = \mathbf{r}(q) = 1$.*

Examples from [DL] show that it is difficult to determine precisely what happens to the rank of a Hermitian symmetric polynomial q under multiplication.

The crucial information in the statement of the next proposition is that the integers are non-zero. We have seen already that we can obtain $(2, 0)$ for the

signature pair of a product when one of the factors has signature pair $(A, 0)$. What happens if we insist that neither factor has signature pair $(k, 0)$, in other words, that neither factor is a squared norm? Remarkably, we can still get $(2, 0)$. In fact we can get any pair except $(0, 0)$ (obviously), $(1, 0)$, or $(0, 1)$.

Proposition 9.3 (DL). *Assume $N \geq 2$. Then there exist Hermitian symmetric polynomials r_1 and r_2 such that $\mathbf{s}(r_j) = (A_j, B_j)$, none of the four integers A_j or B_j is zero, and such that $\mathbf{s}(r_1 r_2) = (N, 0)$.*

In other words, given an integer N at least 2, we can find a squared norm with rank N which can be factored such that neither factor is a squared norm.

10. BIBLIOGRAPHY

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